Table 7.2 Laplace Transform of some standard signals

$x(t)$	$X(s)$
$\delta(t)$	1
1	$\frac{1}{s}$
t	$\frac{1}{s^2}$
t^n	$\frac{n!}{s^{n+1}}$
e^{-at}	$\frac{1}{s+a}$
te^{-at}	$\frac{1}{(s+a)^2}$
$t^n e^{-at}$	$\frac{n!}{(s+a)^{n+1}}$
$\sin\Omega_0 t$	$\frac{\Omega_0}{s^2+\Omega_0^2}$
$\cos\Omega_0 t$	$\frac{s}{s^2+\Omega_0^2}$
$e^{-at}\sin\Omega_0 t$	$\frac{\Omega_0}{(s+a)^2+\Omega_0^2}$
$e^{-at}\cos\Omega_0 t$	$\frac{s+a}{(s+a)^2+\Omega_0^2}$

Solved Problem 7.27 A signal has Laplace transform

$$X(s)=\frac{s+2}{s^2+4s+5}$$

Find Laplace transforms $Y(s)$, of the following signals

(a) $y_1(t)=tx(t)$ (b) $y_2(t)=e^{-t}x(t)$ (c) $y_3(t)=x(t)*x(t)$

Solution:

a) Given $X(s)=\dfrac{s+2}{s^2+4s+5}$

By using differentiation in s-domain property, we have

$$Y_1(s)=L[y_1(t)]=L[tx(t)]=\frac{-d}{ds}X(s)$$

$$=\frac{-d}{ds}\left[\frac{s+2}{s^2+4s+5}\right]$$

$$= -\left[\frac{(s^2+4s+5)-(s+2)(2s+4)}{(s^2+4s+5)^2}\right]$$

$$= -\left[\frac{(s^2+4s+5-2s^2-8s-8)}{(s^2+4s+5)^2}\right] = \frac{s^2+4s+3}{(s^2+4s+5)^2}$$

b) $y_2(t) = e^{-t}x(t)$

We have frequency shifting property, we know that

$$L[e^{-at}x(t)] = X(s+a)$$

$$L[e^{-t}x(t)] = X(s+1)$$

$$Y_2(s) = \frac{(s+1)+2}{(s+1)^2+4(s+1)+5} = \frac{s+3}{s^2+6s+10}$$

c) By using convolution property, we have

$$L[x(t)*x(t)] = X(s)\cdot X(s)$$

$$= [X(s)]^2$$

$$= \left[\frac{s+2}{s^2+4s+5}\right]^2$$

Solved Problem 7.28 If $x(t)$ is even function, prove that $X(s) = X(-s)$ and if $x(t)$ is odd, prove that $X(s) = -X(-s)$

Solution:

$$X(s) = F[x(t)] = \int_{-\infty}^{\infty} x(t)e^{-st}\,dt$$

$$X(-s) = \int_{-\infty}^{\infty} x(t)e^{st}\,dt$$

Let $t = -p$ then $dt = -dp$

$$X(-s) = \int_{-\infty}^{\infty} x(-p)e^{-sp}\,(dp)$$

$$= \int_{-\infty}^{\infty} x(-p)e^{-sp}\,dp$$

$$= \int_{-\infty}^{\infty} x(-t)e^{-st}\,dt$$

Since $x(t) = x(-t)$ for even

$$X(-s) = \int_{-\infty}^{\infty} x(t)e^{-st}\,dt$$

$$= X(s)$$

We have

$$X(-s) = \int_{-\infty}^{\infty} x(-t)e^{-st}\,dt$$

For odd signal $x(t) = -x(-t)$. Therefore

$$X(-s) = -\int_{-\infty}^{\infty} x(t)e^{-st}\,dt$$

$$= -X(s)$$

Practice Problem 7.8

A signal has Laplace transform

$$X(s) = \frac{2}{s^2 + 4s + 3}$$

Find Laplace transforms of the following signals

(a) $y(t) = x(2t)$ (b) $y_2(t) = t^2 x(t)$

7.6 Inversion of Unilateral Laplace Transform

The direct method of finding Laplace transform using Equation

$$x(t) = \frac{1}{2\pi j} \int_{\sigma - j\infty}^{\sigma + j\infty} X(s)e^{st}\,ds \tag{7.48}$$

requires an understanding of complex variable. There are other methods to find inverse Laplace transform that uses one-to-one relationship between a signal and its unilateral Laplace transform. One such method is using partial fraction expansion.

Let

$$X(s) = \frac{N(s)}{D(s)} = \frac{b_M s^M + b_{M-1} s^{M-1} + \ldots + b_1 s + b_0}{s^N + a_{N-1} s^{N-1} + \ldots + a_1 s + a_0} \tag{7.49}$$

where $M < N$.

To find partial fraction expansion, first we find the roots of the denominator polynomial, say $p_1, p_2 \ldots p_k$. These roots are the poles of $X(s)$. Now we can express $X(s)$ as

$$X(s) = \frac{N(s)}{D(s)} = \frac{b_M s^M + b_{M-1} s^{M-1} + \ldots + b_1 s + b_0}{\prod_{k=1}^{N} (s - p_k)} \tag{7.50}$$

7.6.1 Distinct poles

If all the poles p_k are distinct, then we may write $X(s)$ as a sum of single pole terms given by

$$X(s) = \frac{C_1}{s - p_1} + \frac{C_2}{s - p_2} + \ldots + \frac{C_i}{s - p_i} + \ldots + \frac{C_k}{s - p_k} \tag{7.51}$$

Multiply both sides of Eq. (7.51) by $(s - p_i)$ and substitute $s = p_i$

$$\begin{aligned}(s - p_i) X(s)\Big|_{s=p_i} &= \Big|(s - p_i)\frac{C_1}{s - p_1} + (s - p_i)\frac{C_2}{s - p_2} + \ldots + (s - p_i)\frac{C_i}{(s - p_i)} \\ &\quad + \ldots + (s - p_i)\frac{C_k}{(s - p_k)}\Big|_{s=p_i} \\ &= C_i\end{aligned}$$

That is, the coefficients C_i in Eq. (7.51) can be found by using

$$C_i = (s - p_i) X(s)\Big|_{s=p_i} \tag{7.52}$$

For example if

$$X(s) = \frac{2}{(s-1)(s-2)}$$

then we can write

$$X(s) = \frac{C_1}{s-1} + \frac{C_2}{s-2}$$

where

$$C_1 = (s-1)X(s)\Big|_{s=1} = (s-1)\frac{2}{(s-1)(s-2)}\Big|_{s=1} = -2$$

$$C_2 = (s-2)X(s)\Big|_{s=2} = (s-2)\frac{2}{(s-1)(s-2)}\Big|_{s=2} = 2$$

Therefore

$$X(s) = \frac{-2}{s-1} + \frac{2}{s-2}$$

7.6.2 Multiple poles

In the case of multiple roots say p_k repeats ℓ times, then the expansion of Eq. (7.50) must include terms

$$\frac{C_{1k}}{s-p_k} + \frac{C_{2k}}{(s-p_k)^2} + \ldots + \frac{C_{ik}}{(s-p_k)^i} + \ldots + \frac{C_{\ell k}}{(s-p_k)^\ell} \quad (7.53)$$

The coefficient C_{ik} are evaluated by multiplying both sides of Eq. (7.53) by $(s-p_k)^\ell$, differentiating $(\ell - i)$ times and then evaluating the resultant equation at $s = p_k$. Thus

$$C_{\ell k} = (s-p_k)^\ell X(s)\Big|_{s=p_k}$$

$$C_{\ell-1k} = \frac{d}{ds}[(s-p_k)^\ell X(s)]\Big|_{s=p_k}$$

$$\vdots$$

$$C_{ik} = \frac{1}{(\ell-i)!}\frac{d^{\ell-i}}{ds^{\ell-i}}\left[(s-p_k)^\ell X(s)\right]\Big|_{s=p_k} \quad (7.54)$$

$$\text{Let } X(s) = \frac{2s+1}{(s+2)^3}$$

$$= \frac{C_{11}}{s+2} + \frac{C_{21}}{(s+2)^2} + \frac{C_{31}}{(s+2)^3}$$

By using Eq. (7.54), we have

$$C_{31} = (s+2)^3 X(s)\Big|_{s=-2}$$

$$= \cancel{(s+2^3)} \; \frac{(2s+1)}{\cancel{(s+2^3)}}\Bigg|_{s=-2} = 2(-2)+1 = -3$$

$$C_{21} = \frac{d}{ds}\left[(s+2)^3 X(s)\right]\Big|_{s=-2}$$

$$= \frac{d}{ds}\left[\cancel{(s+2^3)} \; \frac{2s+1}{\cancel{(s+2^3)}}\right]\Bigg|_{s=-2} = 2$$

$$= \frac{1}{(3-1)!}\frac{d^2}{ds^2}\left[\cancel{(s+2^3)} \; \frac{(2s+1)}{\cancel{(s+2^3)}}\right]\Bigg|_{s=-2}$$

$$= \frac{1}{2!}\left[\frac{d^2}{ds^2}(2s+1)\right]\Bigg|_{s=-2} = 0$$

$$\Rightarrow X(s) = \frac{2}{(s+2)^2} - \frac{3}{(s+2)^3}$$

7.6.3 Complex roots

If $X(s)$ has complex poles then partial fraction of the $X(s)$ can be expressed as

$$X(s) = \frac{C_1}{s-p_1} + \frac{C_1^*}{s-p_1^*} \tag{7.55}$$

where C_1^* is complex conjugate of C_1 and p_1^* is complex conjugate of p_1. In otherwords, complex conjugate poles result in complex conjugate coefficients.

Let $X(s) = \dfrac{s}{s^2+2s+2}$

$$= \frac{C_1}{s+1+j1} + \frac{C_2}{s+1-j1}$$

Roots of s^2+2s+2
$s_{1,2} = s^2+2s+2 = 0 \; \frac{-2\pm\sqrt{-4}}{2} = -1 \pm j1$

$$C_1 = (s+1+j1)X(s)\Big|_{s=-1-j1}$$

$$= (s+1+j1)\frac{s}{(s+1+j1)(s+1-j1)}\Big|_{s=-1-j1} = \frac{-1-j1}{-j2}$$

$$= \frac{1+j1}{j2} = 0.5 - j0.5$$

$$C_2 = (s+1-j1)\frac{s}{(s+1+j1)(s+1-j1)}\Big|_{s=-1+j1} = \frac{-1+j1}{j2} = 0.5 + j0.5$$

$$\Rightarrow \quad C_2 = C_1^*$$

Therefore complex conjugate poles results in complex conjugate coefficients.

Alternative method

The above procedure of finding partial fraction involves manipulation of complex numbers. When the denominator has complex roots, the complex pair can be expressed as a single quadratic factor instead of having first - order partial fractions. That is, if $s = -a \pm jb$, we can retain the quadratic factor

$$D(s) = (s+a+jb)(s+a-jb)$$
$$= (s+a)^2 + b^2$$

Let us consider

$$X(s) = \frac{N(s)}{D(s)} = \frac{2s+3}{(s+2)(s^2+4s+8)}$$

The roots of the denominator $D(s)$ are $-2; -2 \pm j2$.
Instead of expressing $X(s)$ with first - order partial fractions as

$$X(s) = \frac{A}{s+2} + \frac{B}{s+2+j2} + \frac{B^*}{s+2-j2}$$

we can retain the quadratic factor s^2+4s+8 and express $X(s)$ as

$$X(s) = \frac{2s+3}{(s+2)(s^2+4s+8)} = \frac{A}{s+2} + \frac{Bs+C}{s^2+4s+8}$$
$$= \frac{A(s^2+4s+8)+(s+2)(Bs+C)}{(s+2)(s^2+4s+8)}$$

Equating numerator we get

$$2s+3 = A(s^2+4s+8)+(s+2)(Bs+C)$$
$$= (A+B)s^2 + (4A+2B+C)s + 8A + 2C$$

Now equating the coefficients of s^2, s and constant yields

$$A+B=0$$

$$4A+2B+C=2$$

$$8A+2C=3$$

$$\Rightarrow A=\frac{-1}{4}; B=\frac{1}{4}; C=\frac{5}{2}$$

$$X(s)=\frac{-1}{4(s+2)}+\frac{\frac{1}{4}s+\frac{5}{2}}{s^2+4s+8}$$

$$\boxed{\begin{aligned} 4A+2B+C &= 2\times 2\\ 8A+2C &= 3\\ \hline 8A+4B+2C &= 4\\ 8A+2C &= 3\\ \hline 4B &= 1\\ B &= \frac{1}{4}\\ A &= \frac{-1}{4}\\ C &= 5/2 \end{aligned}}$$

Solved Problem 7.29 Find the inverse Laplace transform of the following

(i) $X(s)=\dfrac{s}{s^2+5s+6}$ (ii) $X(s)=\dfrac{3s^2+8s+6}{(s+2)(s^2+2s+1)}$

(iii) $X(s)=\dfrac{2s+1}{(s+1)(s^2+2s+2)}$

Solution:

(i) $X(s)=\dfrac{s}{s^2+5s+6}$

$$=\frac{s}{(s+2)(s+3)}$$

$$=\frac{A}{s+2}+\frac{B}{s+3}$$

$$A=\cancel{(s+2)}\frac{s}{\cancel{(s+2)}(s+3)}\Bigg|_{s=-2}$$

$$=\frac{(-2)}{(-2+3)}=-2$$

$$B=\cancel{(s+3)}\frac{s}{(s+2)\cancel{(s+3)}}\Bigg|_{s=-3}$$

$$=\frac{-3}{(-3+2)}=3$$

$$X(s)=\frac{-2}{(s+2)}+\frac{3}{s+3}$$

Taking inverse Laplace transform we get

$$x(t) = -2e^{-2t}u(t) + 3e^{-3t}u(t)$$

(ii) Given

$$X(s) = \frac{3s^2+8s+6}{(s+2)(s^2+2s+1)} = \frac{3s^2+8s+6}{(s+2)(s+1)^2}$$

$$= \frac{A}{s+2} + \frac{B}{s+1} + \frac{C}{(s+1)^2}$$

$$A = \cancel{(s+2)} \left.\frac{3s^2+8s+6}{\cancel{(s+2)}\,(s+1)^2}\right|_{s=-2}$$

$$= \frac{3(-2)^2+8(-2)+6}{(-2+1)^2}$$

$$= \frac{3(4)-16+6}{1} = 2$$

$$B = \frac{1}{1!}\frac{d}{ds}\left[\cancel{(s+1)^2} \frac{3s^2+8s+6}{(s+2)\,\cancel{(s+1)^2}}\right]$$

$$= \left.\frac{(s+2)(6s+8)-(3s^2+8s+6)}{(s+2)^2}\right|_{s=-1}$$

$$= \frac{(-1+2)(-6+8)-(3-8+6)}{(-1+2)^2}$$

$$= 1$$

$$C = \cancel{(s+1)^2} \left.\frac{3s^2+8s+6}{\cancel{(s+1)^2}\,(s+2)}\right|_{s=-1}$$

$$= \frac{3(-1)^2+8(-1)+6}{(-1+2)} = 1$$

$$X(s) = \frac{2}{s+2} + \frac{1}{s+1} + \frac{1}{(s+1)^2}$$

$$\boxed{\begin{aligned} L^{-1}\left[\frac{1}{s+2}\right] &= e^{-2t}u(t) \\ L^{-1}\left[\frac{1}{s+1}\right] &= e^{-t}u(t) \\ L^{-1}\left[\frac{1}{(s+2)^2}\right] &= te^{-t}u(t) \end{aligned}}$$

Taking inverse Laplace transform we get

$$x(t) = 2e^{-2t}u(t) + e^{-t}u(t) + te^{-t}u(t)$$

(iii) $X(s) = \dfrac{2s+1}{(s+1)(s^2+2s+2)}$

$$= \frac{A}{s+1} + \frac{B}{s-(-1+j1)} + \frac{B^*}{s-(-1-j1)}$$

$$\boxed{\begin{aligned}&\text{Roots of } s^2+2s+2=0 \text{ are} \\ &\frac{-2\pm\sqrt{4-8}}{2} = \frac{-2\pm j2}{2} \\ &= -1\pm j1\end{aligned}}$$

$$A = (s+1)\frac{(2s+1)}{(s+1)(s^2+2s+2)}\bigg|_{s=-1}$$

$$= \frac{2(-1)+1}{(-1)^2+2(-1)+2} = \frac{-1}{1} = -1$$

$$B = (s+1-j)\frac{2s+1}{(s+1)(s+1+j)(s+1-j)}\bigg|_{s=-1+j}$$

$$= \frac{2(-1+j)+1}{(-1+j+1)(-1+j+1+j)}$$

$$= \frac{-1+2j}{j(2j)} = \frac{-1}{2}(-1+2j) = 0.5-j$$

$$\Rightarrow X(s) = \frac{-1}{s+1} + \frac{0.5-j}{s-(-1+j1)} + \frac{0.5+j}{s-(-1-j1)}$$

Taking inverse Laplace-transform we get

$$\begin{aligned}x(t) &= -e^{-t}u(t) + (0.5-j)e^{(-1+j1)t}u(t) + (0.5+j)e^{(-1-j1)t}u(t) \\ &= -e^{-t}u(t) + (0.5-j)e^{-t}e^{jt}u(t) + (0.5+j)e^{-t}e^{-jt}u(t) \\ &= -e^{-t}u(t) + 0.5e^{-t}e^{jt}u(t) - je^{-t}e^{jt}u(t) + 0.5e^{-t}e^{-jt}u(t) \\ &\quad + je^{-t}e^{-jt}u(t) \\ &= -e^{-t}u(t) + 0.5e^{-t}(e^{jt}+e^{-jt})u(t) - je^{-t}(e^{jt}-e^{-jt})u(t) \\ &= -e^{-t}u(t) + e^{-t}\cos t\, u(t) + 2e^{-t}\sin t\, u(t) \\ &= -e^{-t}u(t) + e^{-t}(\cos t + 2\sin t)u(t)\end{aligned}$$

$$\boxed{\begin{aligned}\frac{e^{jt}+e^{-jt}}{2} &= \cos t \\ e^{jt}-e^{-jt} &= 2j\sin t\end{aligned}}$$

(or)

$$X(s) = \frac{2s+1}{(s+1)(s^2+2s+2)} = \frac{A}{s+1} + \frac{Bs+C}{s^2+2s+2}$$

$$\begin{aligned}\frac{2s+1}{(s+1)(s^2+2s+2)} &= \frac{A}{s+1} + \frac{Bs+C}{s^2+2s+2} \\ &= \frac{A(s^2+2s+2)+(Bs+C)(s+1)}{(s+1)(s^2+2s+2)} \\ &= \frac{(A+B)s^2+(2A+B+C)s+(2A+C)}{(s+1)(s^2+2s+2)}\end{aligned}$$

Comparing the numerators of LHS and RHS we get

$$A+B=0$$
$$2A+B+C=2$$
$$2A+C=1$$
$$\Rightarrow \quad B=1$$
$$A=-1$$
$$C=3$$

$$L^{-1}\left[\frac{s+a}{(s+a)^2+b^2}\right]=e^{-at}\cos bt$$
$$L^{-1}\left[\frac{b}{(s+a)^2+b^2}\right]=e^{-at}\sin bt$$

$$X(s)=\frac{-1}{s+1}+\frac{s+3}{s^2+2s+2}$$

$$=\frac{-1}{s+1}+\frac{s+3}{(s+1)^2+1}$$
$$=\frac{-1}{s+1}+\frac{s+1}{(s+1)^2+1}+\frac{2}{(s+1)^2+1^2}$$

Taking inverse Laplace transform

$$x(t)=-e^{-t}u(t)+e^{-t}\cos tu(t)+2e^{-t}\sin tu(t)$$

Practice Problem 7.9

Find the inverse Laplace transform of the following functions.

(i) $X(s)=\dfrac{s+2}{s(s+1)^2}$ (ii) $X(s)=\dfrac{s+1}{s(s+2)^2(s^2+4s+5)}$

(iii) $X(s)=\dfrac{4s^2+8s+10}{(s+2)(s^2+2s+5)}$ (iv) $X(s)=\dfrac{3s^2+22s+27}{(s^2+3s+2)(s^2+2s+5)}$

JNTU May 2005, Nov 2005

Solved Problem 7.30 Use the convolution theorem of Laplace transform to find $y(t)=x_1(t)*x_2(t)$ where $x_1(t)$ and $x_2(t)$ are given below.

(a) $x_1(t)=e^{-3t}u(t)$ and $x_2(t)=u(t-2)$

(b) $x_1(t)=\cos(4t)u(t)$ and $x_2(t)=\sin(2t)u(t)$

Solution:

From convolution property we have, $L[x_1(t)*x_2(t)]=X_1(s)X_2(s)$

(a) $x_1(t) = e^{-3t}u(t)$

$$X_1(s) = L[e^{-3t}u(t)] = \frac{1}{s+3}$$

$$x_2(t) = u(t-2)$$

$$X_2(s) = L[u(t-2)] = \frac{e^{-2s}}{s}$$

$$L[u(t)] = \frac{1}{s}$$

Using time shifting property

$$L[u(t-2)] = \frac{e^{-2s}}{s}$$

$$L[x_1(t) * x_2(t)] = X_1(s)X_2(s) = \frac{e^{-2s}}{s(s+3)}$$

Let

$$Y(s) = \frac{e^{-2s}}{s(s+3)} = e^{-2s}Y_1(s)$$

where

$$Y_1(s) = \frac{1}{s(s+3)}$$

$$Y_1(s) = \frac{A}{s} + \frac{A}{s+3}$$

$$A = s\left.\frac{1}{s(s+3)}\right|_{s=0} = \frac{1}{3}$$

$$B = (s+3)\left.\frac{1}{s(s+3)}\right|_{s=-3} = \frac{-1}{3}$$

$$Y_1(s) = \frac{1}{3s} - \frac{1}{3(s+3)}$$

$$y_1(t) = \left[\frac{1}{3} - \frac{1}{3}e^{-3t}\right]u(t)$$

By using time shifting property we can write

$$y(t) = L^{-1}[e^{-2s}Y_1(s)] = y_1(t-2)$$

$$\Rightarrow \quad y(t) = \frac{1}{3}u(t-2) - \frac{1}{3}e^{-3(t-2)}u(t-2)$$

(b) $x_1(t) = \cos(4t)u(t)$; $X_1(s) = \frac{s}{s^2+16}$

$$L[\cos(\Omega_0 t)u(t)] = \frac{s}{s^2+\Omega_0^2}$$

$$L[\sin(\Omega_0 t)u(t)] = \frac{\Omega_0}{s^2+\Omega_0^2}$$

$$x_2(t) = \sin(2t)u(t); X_2(s) = \frac{2}{s^2+4}$$

$$L[x_1(t) * x_2(t)] = X_1(s)X_2(s)$$
$$= \frac{s}{s^2+16} \cdot \frac{2}{s^2+4}$$
$$= \frac{2s}{(s^2+16)(s^2+4)}$$

$$\text{Let } Y(s) = \frac{2s}{(s^2+16)(s^2+4)}$$
$$= \frac{As+B}{s^2+16} + \frac{Cs+D}{s^2+4}$$
$$2s = (As+B)(s^2+4) + (Cs+D)(s^2+16)$$
$$\Rightarrow \quad = As^3 + Bs^2 + 4As + 4B + Cs^3 + Ds^2 + 16Cs + 16D$$
$$= (A+C)s^3 + (B+D)s^2 + (4A+16C)s + 4B + 16D$$

Comparing coefficients of s^3, s^2, s and constant terms we get

$$A + C = 0$$
$$B + D = 0$$
$$4A + 16C = 2$$
$$4B + 16D = 0$$

Solving for A, B, C and D we get $A = \frac{-1}{6}$ $B = 0$ $C = \frac{1}{6}$ and $D = 0$

$$\Rightarrow \quad X(s) = \frac{-1}{6}\left[\frac{s}{s^2+16} - \frac{s}{s^2+4}\right]$$
$$x(t) = \frac{-1}{6}[\cos 4t - \cos 2t]u(t)$$

Practice problem 7.10

Use the convolution theorem of Laplace transform to find $y(t) = x_1(t) * x_2(t)$ where $x_1(t)$ and $x_2(t)$ are given below

(i) $x_1(t) = e^{-3t}u(t)$ and $x_2(t) = e^{-2t}u(t)$

(ii) $x_1(t) = tu(t)$ and $x_2(t) = \delta(t-1) - \delta(t-4)$

7.7 Inversion of the Bilateral Laplace Transform

So far we discussed the inversion of unilateral Laplace transform, which can be used only for causal signals. If $x(t)$ is a non-causal signal then to find $x(t)$ from $X(s)$., we must know the region of convergence of $X(s)$. The location of pole along with ROC determines, whether a given pole corresponds to a positive or negative time portion of $x(t)$. From solved problem (7.1) we find that if the ROC is Re $(s) > a$, then the signal $x(t)$ is a causal signal. On the other hand if the ROC is Re $(s) < a$, then the signal $x(t)$ is a non-causal signal. In other words, if a pole of $X(s)$ lies to the left of the ROC, then this pole gives rise to a causal signal and if a pole of $X(s)$ lies to the right of the ROC, then this poles gives a non-causal signal. For example let us consider a two-sided signal $x(t)$ whole Laplace transform is given by

$$X(s) = \frac{2}{(s+3)(s+2)} \quad \text{ROC:} -3 < \text{Re}(s) < -2$$

The ROC is shown in Fig. 7.13.

Here, the pole $s = -3$ lies to the left of ROC, hence the pole gives rise to a causal signal. The poles $s = -2$ lies to the right of ROC, hence the pole gives rise to a non causal signal. We now expand $X(s)$ in partial fraction to obtain.

$$X(s) = \frac{2}{s+2} - \frac{2}{s+3}$$

$$\text{ROC:} -3 < \text{Re}(s) < -2$$

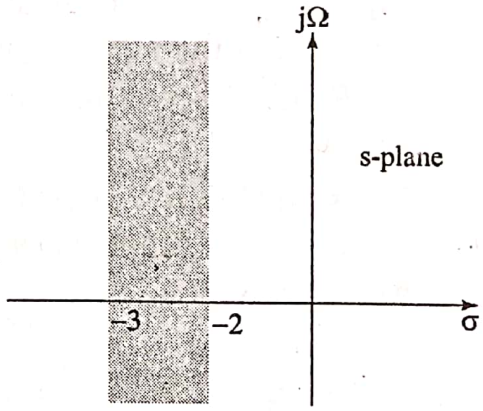

Fig. 7.13

Now

$$L^{-1}\left[\frac{2}{s+2}\right] = -2e^{-2t}u(-t)$$

and

$$L^{-1}\left[\frac{2}{s+3}\right] = 2e^{-3t}u(t)$$

Therefore

$$x(t) = -2e^{-2t}u(-t) - 2e^{-3t}u(t)$$

Solved Problem 7.31 Find the inverse Laplace transform of

$$X(s) = \frac{2}{(s+4)(s-1)} \quad \text{if the region of convergence is}$$

(b) $\text{Re}(s) > 1$
(c) $\text{Re}(s) < -4$

Solution:

Given

$$X(s) = \frac{2}{(s+4)(s-1)}$$
$$= \frac{A}{s+4} + \frac{B}{s-1}$$
$$A = \cancel{(s+4)} \frac{2}{\cancel{(s+4)}(s-1)}\bigg|_{s=-4} = \frac{-2}{5}$$
$$B = \cancel{(s-1)} \frac{2}{(s+4)\cancel{(s-1)}}\bigg|_{s=1} = \frac{2}{5}$$
$$\Rightarrow X(s) = \frac{-2}{5}\frac{1}{s+4} + \frac{2}{5(s-1)}$$

(a) The $X(s)$ has poles at -4 and 1. The strip of ROC is $-4 < \text{Re}(s) < 1$ as shown in Fig. 7.14. The pole at -4, which is at the left of the strip of ROC, corresponds to the causal signal and the pole at 1 to the right of the strip of ROC corresponds to anticausal signal. Therefore

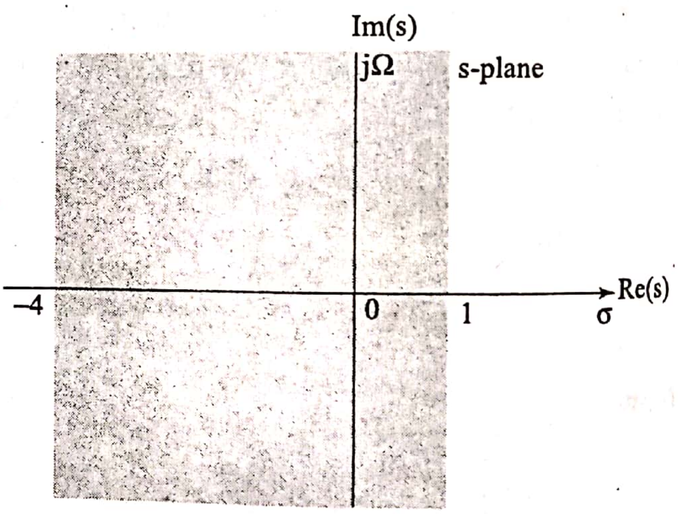

Fig. 7.14

$$x(t) = \frac{-2}{5}e^{-4t}u(t) - \frac{2}{5}e^{t}u(-t)$$

(b) The ROC is $\text{Re}(s) > 1$.

Both poles lies to the left of the ROC, so both poles correspond to causal signal. Therefore

$$x(t) = \frac{-2}{5}e^{-4t}u(t) + \frac{2}{5}e^{t}u(t)$$

(c) The ROC is $\text{Re}(s) < -4$

Both poles lie to the right of the ROC. So both poles correspond to anti-causal signals. Therefore

$$x(t) = \frac{2}{5}e^{-4t}u(-t) - \frac{-2}{5}e^{t}u(-t)$$

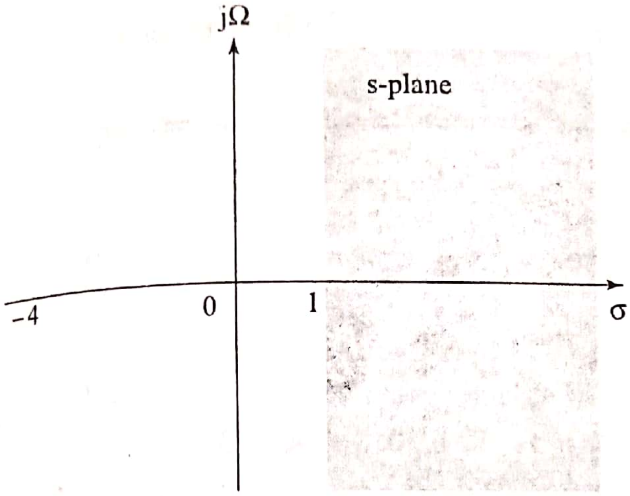

Fig. 7.15

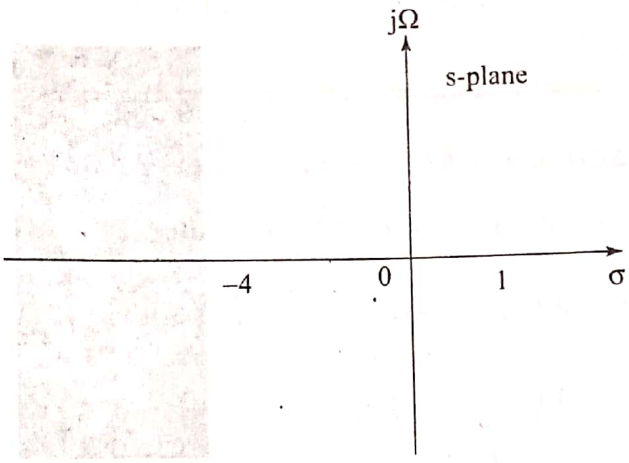

Fig. 7.16

Solved Problem 7.32 Find the signal whose bilateral transform is

$$X(s) = \frac{1}{(s+5)(s+1)} \qquad -5 < \text{Re}(s) < -1$$

Solution:

$$X(s) = \frac{1}{(s+5)(s+1)}$$

$$= \frac{A}{s+5} + \frac{B}{s+1}$$

$$A = (s+5)\frac{1}{(s+5)(s+1)}\bigg|_{s=-5} = \frac{-1}{4}$$

$$B = (s+1)\frac{1}{(s+5)(s+1)}\bigg|_{s=-1} = \frac{1}{4}$$

$$X(s) = \frac{-1}{4}\cdot\frac{1}{s+5} + \frac{1}{4}\cdot\frac{1}{s+1}$$

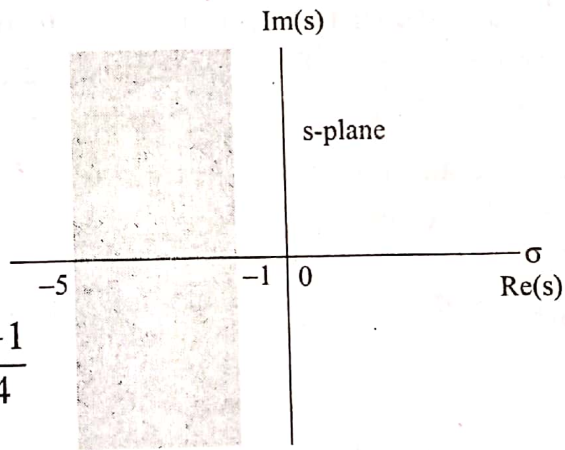

Fig. 7.17

ROC is $-5 < \text{Re}(s) < -1$ shown in Fig. 7.17

The pole -5 is left to the region of convergence so this correspond to causal signal and the pole -1 is right of the ROC so this corresponds to anticausal signal.

$$x(t) = \frac{-1}{4}e^{-5t}u(t) - \frac{1}{4}e^{-t}u(-t)$$